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IMPLEMENTING ARROW-DEBREU EQUILIBRIA  
BY CONTINUOUS TRADING OF FEW LONG-LIVED SECURITIES

by

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and

Chi-fu Huang\*\*

August 1983

MIT Sloan School of Management WP #1501-83

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## Abstract

A two-period ( $0$  and  $T$ ) Arrow-Debreu economy is set up with a general model of uncertainty. We suppose that an equilibrium exists for this economy. The Arrow-Debreu economy is placed in a Radner [31] setting; agents may trade claims continuously during  $[0, T]$ . Under appropriate conditions it is possible to implement the original Arrow-Debreu equilibrium, which may have an infinite dimensional commodity space, in a Radner economy which has only a finite number of securities. This is done by opening the "right" set of securities markets, a set which effectively completes markets for the continuous trading Radner economy.

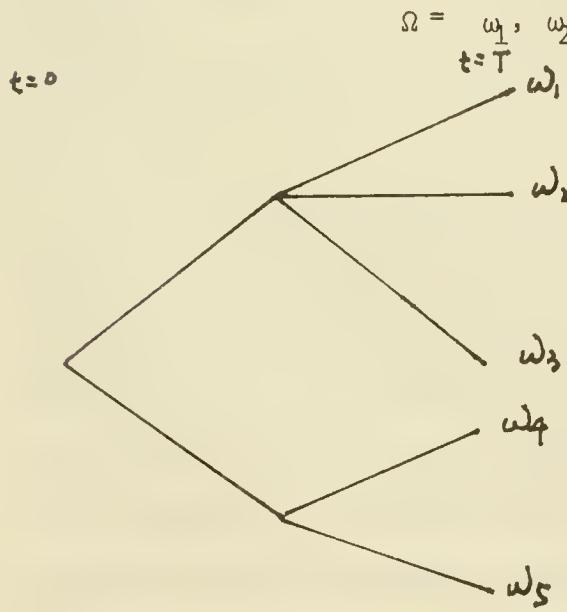


## 1.0 Introduction

Event tree A of Figure 1 depicts a simple information structure.

Let's momentarily consider an exchange economy with endowments of and preferences for (random) time T consumption which depends on the state  $\omega \in \Omega$  chosen by nature from the final nodes of this event tree. A competitive equilibrium will exist under standard assumptions (Debreu [3], Chapter 7) including markets for securities whose time T consumption payoff vectors span  $\mathbb{R}^5$ . This entails at least five security markets, while intuition suggests that, with the ability to learn information and trade during  $[0, T]$ , only three securities which are always available for trading (long-lived securities [9]) might be enough to effectively complete markets. (This is the maximum number of branches leaving any node in the tree. The reasoning is given by Kreps [10], and in an alternative form later in this paper.) One major purpose of this paper is to verify this intuition for a general class of information structures, including those which cannot be represented by

Event Tree A



Event Tree B

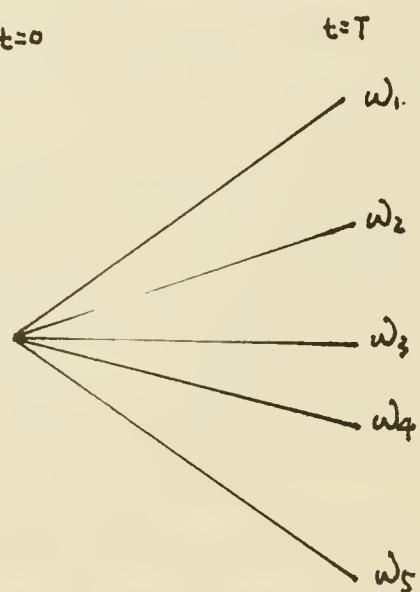


Figure 1      Event Trees



event trees (such as the filtrations generated by continuous-time stochastic process.) In some cases, where an Arrow-Debreu style equilibrium would call for an infinite number of securities, we show how a continuous trading Radner [16] equilibrium of plans, prices and price expectations can implement the same Arrow-Debreu consumption allocations with only a finite number of long-lived securities.

A comparision of Event Trees A and B, intended to correspond to the same Arrow-Debreu economy, obviates the role of the information structure in determining the number of long-lived securities required to "span" the consumption space, or the spanning number. (This term is later given a precise meaning.) Since all uncertainty is resolved at once in Event Tree B, the spanning number is five (instead of three for Event Tree A). Roughly speaking, the maximum number of "dimensions of uncertainty" which could be resolved at any one time is the key determining property. This vague concept actually takes a precise form as the martingale multiplicity of the information structure (See Appendix.) A key result of this paper is that the spanning number is the martingale multiplicity plus one. The "plus one" is no mystery; in addition to spanning uncertainty, agents must have the ability to transfer purchasing power across time.

The notion that certain securities are redundant because their payoffs can be replicated by trading other securities over time, yielding arbitrage pricing relationships among securities, was dramatized in the Black-Scholes [1] option pricing formula.<sup>1</sup> Provided the equilibrium price process for one security happens to be a geometric Brownian Motion, and for another is a (deterministic) exponential of time, then any contingent claim whose payoff depends (measurably) on the path taken by the underlying Brownian Motion, such as a call option on the risky



security, is redundant and priced by arbitrage. This discovery curiously preceded an understanding of its simpler logical antecedents, such as corresponding results for event tree information structures. Only in the past few years have the implications of the spanning properties of price processes (e.g. [10]), the connection between martingale theory and equilibrium price process (e.g. [5]), and the mathematical machinery for continuous security trading ([6]) been formalized.

In all of the above mentioned literature, the takeoff point is a given set of security price processes (implicitly embedded in a Radner equilibrium). Here we begin more primitively with a given Arrow-Debreu equilibrium, one in which trading over time is not of concern since markets are complete at time zero. From that point we construct the consumption payoffs and price processes for a set of long-lived securities in such a way that agents may be allocated trading strategies allowing them to consume their original Arrow-Debreu allocations within a Radner style equilibrium. In short, we implement a given Arrow-Debreu equilibrium by continuous trading of a set of long-lived securities which is typically much smaller in number than the dimension of the consumption space.

The paper unfolds in the following order. First we describe the economy (Section 2) and an Arrow-Debreu equilibrium for it (Section 3). Section 4 provides a constructive proof of a Radner equilibrium which implements a given Arrow-Debreu equilibrium under stated conditions, based on a martingale representation technique. Section 5 characterizes the spanning number in terms of the martingale multiplicity. Section 6 discusses the continuous trading machinery, some generalizations, and two examples of the model. Section 7 adds concluding remarks.



## 2. The Economy

Uncertainty in our economy is modeled as a complete probability space  $(\Omega, \mathcal{F}, P)$ . The set  $\Omega$  constitutes all possible states of the world which could exist at a terminal date  $T > 0$ . The tribe  $\mathcal{F}$  is the  $\sigma$ -algebra of measurable subsets of  $\Omega$ , or events which agents can make probability assessments of based on the probability measure  $P$ . Events are revealed over time according to a filtration  $\mathcal{F} = \{F_t, t \in [0, T]\}$ , a right-continuous increasing family of sub-tribes of  $\mathcal{F}$ , where  $F_T = \mathcal{F}$  and  $F_0$  is almost trivial (the tribe generated by  $\Omega$  and all of the  $P$ -null sets). The tribe  $F_t$  may be interpreted as the set of all events which could occur at or before time  $t$ .<sup>2</sup>

Each agent in the economy is characterized by the following properties:

- (i) a known endowment of a perishable consumption good at time zero,
- (ii) a random (or state-dependent) endowment of the consumption good at time  $T$ , and
- (iii) preferences over consumption pairs  $(r, x)$ , where  $r$  is time zero consumption and  $x$  is a random variable describing time  $T$  consumption ( $x(\omega)$  in state  $\omega \in \Omega$ ).

We will only consider consumption claims with finite variance. The consumption space is thus formalized as  $V = \mathbb{R} \times L^2(P)$ , where  $L^2(P)$  is the space of (equivalence classes) of square-integrable random variables on  $(\Omega, \mathcal{F}, P)$  with the usual product topology in  $V$  given by the Euclidean and  $L^2$  norms.

The agents are indexed by a finite set  $I = \{1, \dots, I\}$ . The preferences of agent  $i \in I$  are modeled as a complete transitive binary relation  $\succeq_i$  on  $V_i \subseteq V$ , the  $i$ -th agent's consumption set.



The whole economy can then be summarized by the collection

$$\mathcal{E} = (v_i, \hat{v}_i, \succ_i; i \in I),$$

where  $\hat{v}_i = (\hat{r}_i, \hat{x}_i) \in V_i$  is agent  $i$ 's endowment.<sup>3</sup>

### 3. Arrow-Debreu Equilibrium

An Arrow-Debreu equilibrium for  $\mathcal{E}$  is a non-zero linear (price) functional  $\Psi: V \rightarrow \mathbb{R}$  and a set of allocations  $(v_i^* \in V_i; i \in I)$  satisfying, for all  $i \in I$ ,

$$v_i^* \succ_i v' \quad \forall v' \in \hat{V}_i \equiv \left\{ v \in V_i : \Psi(v) \leq \Psi(\hat{v}_i) \right\},$$

$$v_i^* \in \hat{V}_i, \quad \Psi(v) > \Psi(v_i^*) \quad \forall v \succ_i v_i^*$$

$$\sum_{i=1}^I v_i^* \leq \sum_{i=1}^I \hat{v}_i. \tag{3.1}$$

We will assume that preferences are strictly monotonically increasing (in the obvious product ordering on  $V$ ) so that (3.1) holds with equality and  $\Psi$  is a strictly positive linear functional. Since  $V$  is a Hilbert lattice, this then implies that  $\Psi$  is a continuous<sup>4</sup> linear functional on  $V$ , which can therefore be represented as an element  $(a, \xi)$  of  $V$  itself, or:

$$\Psi(r, x) = ar + \int x(w) \xi(w) P(dw) \quad \forall (r, x) \in V.$$

Without loss of generality we can normalize  $\Psi$  by a constant so that  $E(\xi) = 1$ , in order to construct a probability measure  $Q$  on  $(\Omega, \mathcal{F})$  by the relation

$$Q(B) = \int_B \xi(w) P(dw) \quad \forall B \in \mathcal{F}.$$

Equivalently,  $Q$  is defined by the Radon-Nikodym derivative  $\xi = dQ/dP$ .

This leaves the simple representation

$$\Psi(r, x) = ar + E^*(x) \quad \forall (r, x) \in V, \tag{3.2}$$

where  $E^*$  denotes expectation under  $Q$ , so the equilibrium price of any



consumption claim  $x \in L^2(P)$  is simply its expected consumption payoff under  $Q$ . For this reason we call  $Q$  an equilibrium price measure.

For tractability we will want any random variable with finite variance under  $P$  to have finite variance under  $Q$ , and vice versa. A sufficient condition is that  $Q$  and  $P$  are uniformly absolutely continuous, denoted  $Q \approx P$  (Halmos [4], p.100)<sup>5</sup>, or equivalently, that the Radon-Nikodym derivative  $dQ/dP$  is bounded above and below away from zero.

A second regularity condition which comes into play is the separability<sup>6</sup> of  $F$  under  $P$ . Given  $Q \approx P$ , it is then easy to show the separability of  $F$  under  $Q$  by making use of the upper essential bound on  $dQ/dP$ .

Since uniform absolute continuity of two measures implies their equivalence (they have the same null sets), we can use the symbols "a.s." (for "almost surely") indiscriminately in this paper.

#### 4. Radner Equilibrium

A long-lived security is a consumption claim (to some element of  $L^2(P)$ ) available for trade throughout  $[0, T]$ . A price process for long-lived security is a semi-martingale on  $(\Omega, \mathcal{F}, P)$ .<sup>7</sup> In general the number of units of a long-lived security which are held over time defines some stochastic process  $\Theta$  on  $(\Omega, \mathcal{F}, P)$ . We will say  $\Theta$  is an admissible trading process for a long-lived security with price process  $S$  if it meets the regularity conditions:

(i) predictability (defined in Appendix), denoted  $\Theta \in \mathcal{P}$ ,

(ii)  $\Theta \in L_P^2[S] \equiv \left\{ \phi \in \mathcal{P} : E\left(\int_0^T \phi_t^2 d[S]_t\right) < \infty\right\}$ ,

where  $[S]$  denotes the quadratic variation process for  $S$  (Jacod [8]), and

(iii)  $\int \Theta dS$  is well defined as a stochastic integral.<sup>8</sup>

The stochastic integral  $\int_0^t \Theta(s) dS(s)$  is a model of the gains (or



losses) realized up to and including time  $t$  by trading a security with price process  $S$  using the trading process  $\theta$ . Interpreted as a Stieltjes integral, this model is obvious, but the integral is in general well defined only as a stochastic integral. This model, formalized by Harrison and Pliska [6], is discussed further in Section 6, as are the other regularity conditions on  $\theta$ .

Taking  $S = (S_1, \dots, S_N)$  ( $N \leq \infty$ ) as the set of all long-lived security price processes, any corresponding set of trading processes  $\theta = (\theta_1, \dots, \theta_N)$  must meet the accounting identity:

$$\theta(t)^T S(t) = \theta(0)^T S(0) + \int_0^t \theta(s)^T dS(s) \quad \forall t \in [0, T] \quad \text{a.s. (4.1)}$$

meaning the current value of a portfolio must be its initial value plus any gains or losses from trade incurred. [The shorthand notation in (4.1) should be obvious.] We'll adopt the notation  $\Theta(S)$  for the space of trading strategies  $\theta = (\theta_1, \dots, \theta_N)$  meeting the regularity conditions (i)-(ii)-(iii) for each long-lived security and satisfying (4.1).

A Radner equilibrium for  $\mathfrak{E}$  is comprised of:

- (1) a set of long-lived securities claiming  $d = (d_1, \dots, d_N)$  ( $N \leq \infty$ ) with price processes  $S = (S_1, \dots, S_N)$ ,
- (2) a set of trading strategies  $\theta^i \in \Theta(S)$ , one for each agent  $i \in I$   
and
- (3) a price  $a \in \mathbb{R}_+$  for time zero consumption, all satisfying:

Budget Constrained Optimality for each  $i \in I$ :

$(\hat{r}_i - \frac{\theta_i(0)^T S(0)}{a}, \hat{x}_i + \theta_i^T d)$  is  $\lambda_i$ -maximal in the budget set:

$$\left\{ (\hat{r}_i - \frac{\theta_i(0)^T S(0)}{a}, \hat{x}_i + \theta_i^T d) \mid \forall \theta_i : \theta_i \in \Theta(S) \right\} ,$$



and Market Clearing:

$$\sum_{i=1}^I \theta^i(t) = 0 \quad \forall t \in [0, T] \quad \text{a.s.}$$

The space of square-integrable martingales under  $Q$ , denoted  $\mathcal{M}_Q^2$ ; its multiplicity, denoted  $M(\mathcal{M}_Q^2)$ ; and an orthogonal 2-basis for  $\mathcal{M}_Q^2$ , say  $m = (m_1, \dots, m_N)$ ,  $N = M(\mathcal{M}_Q^2) \leq \infty$ , are all defined in the appendix. The following powerful representation theorem plays a major role in demonstrating a Radner equilibrium for  $\mathcal{E}$ .

Theorem 4.1 For any  $x \in \mathcal{M}_Q^2$ , there exists  $\theta = (\theta_1, \dots, \theta_N)$ , where  $\theta_n \in L_Q^2[M_n]$ ,  $1 \leq n \leq N$ , such that

$$x_t = \int_0^t \theta(s)^T dm(s) \quad \forall t \in [0, T] \quad \text{a.s.}$$

Proof: The theorem is an immediate consequence of the definition of  $m$  as an orthogonal 2-basis for  $\mathcal{M}_Q^2$ . (See Jacod [8], Chapter 4). Q.E.D.

We should remark that when  $Q \approx P$ , the spaces  $L_Q^2[m_n]$  and  $L_P^2[m_n]$  are identical, because of the bounds on  $\frac{dQ}{dP}$ . It is also implicit here that a martingale under  $Q$  is a semi-martingale under  $P$ , which can be checked in Jacod [8], Chapter 7, along with the existence of  $\int \theta_n dm_n$  as a stochastic integral on  $(\Omega, \mathcal{F}, P)$  whenever  $\theta_n \in L^2[m_n]$ . We now have the main result.

Theorem 4.2: Suppose  $(\Psi, (v_i^*)_{i \in I})$  is an Arrow-Debreu equilibrium for  $\mathcal{E}$  (W.l.o.g.  $\Psi$  has the representation  $(a, Q)$  given by (3.2)). Provided  $Q \approx P$  and  $\mathcal{F}$  is separable under  $P$ ,  $\mathcal{E}$  has a Radner equilibrium which implements the Arrow-Debreu allocations and is Pareto efficient.



Proof: The proof takes four steps:

1. Specify a set of long-lived securities.
2. Announce a price for time zero consumption and prices for the long-lived securities.
3. Allocate a trading strategy to each agent which generates that agent's Arrow-Debreu allocation and which, collectively, clears markets.
4. Prove that no agent has any incentive to deviate from the allocated trading strategy.

Of course if the Radner equilibrium consumption allocations are the same as the Arrow-Debreu equilibrium allocations, they must be Pareto efficient.

Step 1: Select the following elements of  $L^2(P)$  as the claims of the available long-lived securities:

$$d_0 = 1_{\Omega}$$

$$d_n = m_n(T) \quad 1 \leq n \leq N = M(m_Q^2),$$

where  $1_{\Omega}$  is the indicator on  $\Omega$  and  $m = (m_1, \dots, m_N)$  is an orthogonal 2-basis for  $m_Q^2$ . [Since  $Q \approx P$ , the final values  $m_n(T)$  are elements of  $L^2(P)$ .]

Step 2: Announce the price processes  $S_n(t)$  to be an RCLL version<sup>9</sup> of  $E^*[d_n | F_t]$ ,  $0 \leq n \leq N$ . That is, each long-lived security's current price is the conditional expectation (under  $Q$ ) of its consumption value. There is obviously some forethought here, for the result is  $S_0 \equiv 1$  and  $S_n = m_n$ ,  $1 \leq n \leq N$ , implying the last  $N$  price processes are themselves an orthogonal 2-basis for  $m_Q^2$  suggesting their ability to "span" all consumption claims which are not actually available for trading. The first security serves as a "store-of-value", since its price is constant. We also announce  $a$  as the price of time-zero consumption.

Step 3: For any agent  $i$ ,  $1 \leq i \leq I-1$ , let  $e_i = x_i^* - \hat{x}_i$ . Then the process



$$x_i(t) = E^*(e_i | F_t) - E^*(e_i), \quad t \in [0, T] \quad (4.2)$$

where  $E^*(e_i | F_t)$  is an RCLL version of the conditional expectation,

is an element of  $m_Q^2$  (given  $Q \approx P$ ), which can be reconstructed

via Theorem 4.1, for some  $\Theta_n^i \in L_P^2[S_n]$ ,  $1 \leq n \leq N$ , as

$$x_i(t) = \sum_{n=1}^N \int_0^t \Theta_n^i(s) dS_n(s) \quad \forall t \in [0, T]. \text{ a.s.} \quad (4.3)$$

In order to meet the accounting restriction (4.1), we set the following trading process for the "store-of-value" security

$$\Theta_0^i(t) = \sum_{n=1}^N \int_0^t \Theta_n^i(s) dS_n(s) - \Theta_n^i(t) S_n(t) \quad t \in [0, T]. \quad (4.4)$$

Of course  $\int \Theta_0^i dS_0 \equiv 0$  since  $S_0 \equiv 1$ .

A technical argument showing  $\Theta_0^i \in \mathcal{P}$  is given as Appendix Lemma A.1,

which then implies  $\Theta_0^i \in L_P^2[S_0]$ , and in turn,  $\Theta^i = (\Theta_0^i, \dots, \Theta_N^i) \in \Theta(S)$ .

Substituting (4.4) into (4.3), noting that  $m_n(0) = 0 \quad \forall n$ , then implies

$$\Theta^i(t)^T S(t) = \Theta^i(0)^T S(0) + \int_0^t \Theta^i(s)^T dS(s) \quad \forall t \in [0, T], \text{ a.s.} \quad (4.5)$$

confirming (4.1). Evaluating (4.5) at times  $T$  and  $0$ , using the definitions of  $e_i$

and  $x_i$ , yields:

$$\Theta^i(T)^T S(T) + \hat{x}_i^* = x_i^* \quad \text{a.s.}$$

and

$$\begin{aligned} \Theta^i(0)^T S(0) &= E^*(x_i^* - \hat{x}_i^*) \\ &= \psi(0, x_i^*) - \psi(0, \hat{x}_i^*) = (\hat{r}_i^* - r_i^*) a, \end{aligned}$$

the last line making use of the budget constraint on the Arrow-Debreu allocation for agent  $i$ . Thus by adopting the trading strategy  $\Theta^i$ , and faced with a time-zero consumption price of  $a$ , agent  $i$  can consume precisely  $(r_i^*, x_i^*) = v_i^*$ .

The above construction applies for  $1 \leq i \leq I-1$ ; for the last

agent let  $\Theta^I = -\sum_{i=1}^{I-1} \Theta^i$ . By the Kunita-Watanabe [11] inequality,



$\Theta(S)$  is a linear space, so  $\theta^I \in \Theta(S)$ . Market clearing is obviously met by construction. To complete this step it remains to show that  $\theta^I$  generates the consumption allocation  $(r_I^*, x_I^*) = v_I^*$ , but this is immediate from the linearity of stochastic integrals and market clearing in the Arrow-Debreu equilibrium.

Step 4: We proceed by contradiction. Suppose some agent  $j \in I$  can obtain a preferred allocation  $(r, x) \succ_j (r_j^*, x_j^*)$  by adopting a different trading strategy  $\theta \in \Theta(S)$ . Then the Arrow-Debreu price of  $(r, x)$  must be strictly higher than that of  $(r_j^*, x_j^*)$ , or

$$ar + E^*(x) > a r_j^* + E^*(x_j^*),$$

and substituting the Radner budget constraint for  $r$  and  $x$ ,

$$\begin{aligned} a \hat{r}_j - \Theta(0)^T S(0) + E^* [\hat{x}_j + \Theta(0)^T S(0) + \int_0^T \Theta(t)^T dS(t)] \\ > a r_j^* + E(x_j^*), \end{aligned}$$

or

$$a \hat{r}_j + E^*(\hat{x}_j) > a r_j^* + E^*(x_j^*). \quad (4.6)$$

The last line uses the fact that  $E^* [\int_0^T \Theta(t)^T dS(t)] = 0$  since  $\int \Theta^T dS$

is a Q-martingale for any  $\theta \in \Theta(S)$ , from the fact that  $M_Q^2$

is closed under stochastic integration of this form (see Jacod [8],

Chapter 4). But (4.6) contradicts the Arrow-Debreu budget-constrained optimality of  $(r_j^*, x_j^*)$ . This establishes the theorem.

Q.E.D.

## 5. The Spanning Number of Radner Equilibrium

The key idea of the last proof is that an appropriately selected and priced set of long-lived securities "spans" the entire final-period consumption space in the sense that any  $x \in L^2(P)$  can be represented in the form:

$$E^*(x|F_t) = \Theta(t)^T S(t) = \Theta(0)^T S(0) + \int_0^t \Theta(s)^T dS(s) \quad \forall t \in [0, T], \text{ a.s.} \quad (5.1)$$



where  $S = (S_0, \dots, S_n)$  is the set of  $(N + 1)$  security price processes constructed in the proof and  $\Theta \in \Theta(S)$  is an appropriate trading strategy.<sup>10</sup> As examples in the following section will show, this number  $N + 1$ , the multiplicity of  $M_Q^2$  plus one, can be considerably smaller than the dimension of  $L^2(P)$ . But is this the "smallest number which will work", or the "spanning number", in some sense? To be precise, we will prove the following result. (We still assume  $Q \approx P$  and the separability of  $F$ .)

Proposition 5.1: Suppose long-lived security prices for  $\mathcal{E}$  are square-integrable martingales under  $Q$ , the equilibrium price measure for  $\mathcal{E}$ . Then the minimum number of long-lived securities which completes markets in the sense of (5.1), is  $M(M_Q^2) + 1$ .

Proof: That  $M(M_Q^2) + 1$  is a sufficient number is given by construction in the proof of Theorem 4.2. The remainder of the proof is devoted to showing that at least this number is required.

If  $M(M_Q^2) = \infty$ , we are done. Otherwise, suppose  $S = (S_1, \dots, S_K)$   $K < \infty$ , is a set of square-integrable  $Q$ -martingale security price processes with the representation property (5.1). By the definition of multiplicity, it follows that  $K \geq M(M_Q^2)$ . It remains to show that  $K = M(M_Q^2)$  implies a contradiction, which we now pursue.

Let  $x = k + 1^\top S(T) \in L^2(P)$ , where  $k$  is any real constant and  $1$  is a  $K$ -dimensional vector of ones. If  $S$  has the property (5.1) there exists some  $\Theta \in \Theta(S)$  satisfying (5.1) for this particular  $x$ . Furthermore, since  $S$  is a vector of  $Q$ -martingales,

$$E^*(x|F_t) = k + 1^\top S(t) = k + 1^\top S(0) + \int_0^t 1^\top dS(s) \quad \forall t \in [0, T] \text{ a.s.} \quad (5.2)$$

Since  $\Theta(0)^\top S(0) = E^*(x) = k + 1^\top S(0)$ , equating the right hand sides of (5.1) and (5.2) yields



$$\int_0^t \Theta(s)^T dS(s) = \int_0^t 1^T dS(s) \quad \forall t \in [0, T] \text{ a.s.}$$

Lemma A.2, which is relegated to the appendix due to its technical proof, then implies

$$Q \left\{ \exists t \in [0, T] : \Theta(t) = 1 \right\} > 0.$$

Since  $Q \not\approx P$ , the same event has strictly positive  $P$ -probability, and equating the second members of (5.1) and (5.2) yields

$$P \left\{ \exists t \in [0, T] : 1^T S(t) = 1^T S(t) + k \right\} > 0,$$

an obvious absurdity if  $k \neq 0$ . Q.E.D.

The reader will likely have raised two points by now. First, having shown that the "spanning number" is  $M(\mathcal{M}_Q^2) + 1$  when long-lived security prices are square-integrable martingales under  $Q$ , what do we know about the "spanning number" in general? From the work of Harrison and Kreps [5], we see that a "viable" Radner equilibrium must be of the form of security price processes which are martingales under some probability measure. Their framework, somewhat less general than this, was extended in Huang [7] to a setting much like our own. We leave it to readers to convince themselves that the same conclusions can easily be drawn here. We have chosen to announce prices as martingales under  $Q$ , rather than some other probability measure, as this follows the natural selection of a numeraire claiming one unit of consumption in every state ( $d_0$  in the proof of Theorem 4.2). Other numeraires could be chosen; if a random numeraire is selected then in equilibrium security prices will be martingales under some other probability, say  $\hat{P}$ , and the "spanning number" would be  $M(\mathcal{M}_{\hat{P}}^2) + 1$  (if the proper regularity conditions are adhered to). Does this number differ from  $M(\mathcal{M}_Q^2) + 1$ ; that is, can the martingale multiplicity for the same information structure change under substitution of probability measures? Within the class of



equivalent probability measures (those assigning zero probability to the same events), this seems unlikely. It is certainly not true for event trees. We put off a direct assault on this question to a subsequent paper. We will show later, however, that if the information is generated by a Standard Brownian Motion, then  $M(\mathcal{M}_P^2) = M(\mathcal{M}_Q^2)$ .

The second point which ought to have been raised is the number of securities required to implement an Arrow-Debreu equilibrium in a Radner style model (dropping the requirement for complete markets). For example, with only two agents, a single security which pays the differences between the endowment and the Arrow-Debreu allocation of one of the agents will obviously allow the two to trade to equilibrium at time zero. This is not a very robust regime of markets, of course. By fixing such agent-specific securities, any perturbation of agents' endowments or preferences which preserves Arrow-Debreu prices may preclude an efficient Radner equilibrium. Agents will generally be unable to reach their perturbed Arrow-Debreu allocations without a new set of long-lived securities. A set of long-lived securities which completes markets (in the sense of (5.1)) is contrastingly robust, although our selection still depends endogenously on Arrow-Debreu prices. It remains a formidable challenge to show how markets can be completed by selecting the claims of long-lived securities entirely on the basis of the (exogenous) information structure.<sup>11</sup> There are no economic grounds, of course, precluding the selection of security markets from being an endogenous part of the equilibrium. This would indeed be an interesting problem for future theoretical and empirical research.

## 6.0 Discussion

In this section we discuss some definitional issues, generalizations of the model, and some specific examples.



### 6.1 The gains process and admissible trading strategies

Why is  $L_Q^2[S]$  the "right" restriction on trading strategies

against a security with price process  $S$ ? Why is the stochastic integral  $\int \theta dS$ , for  $\theta \in L_Q^2[S]$  then the appropriate definition of gains from such a strategy?<sup>12</sup> The trading strategies required to represent certain claims in the general case could never be carried out in an actual securities market. No broker or floor trader could move quickly enough.

Following Harrison and Pliska [6], we will say that a predictable trading strategy  $\theta$  is simple, denoted  $\theta \in \Lambda$ , if there is a partition  $0 = \{t_0, t_1, \dots, t_{n-1}, t_n = T\}$  of  $[0, T]$  and bounded random variables  $\{h_i\}_{i=0}^{n-1}$ ,  $h_i \in F_{t_i}$ , satisfying

$$\theta(t) = h_i, \quad t \in (t_i, t_{i+1}].$$

A simple trading strategy  $\theta$ , in words, is one which is piecewise constant and for which  $\theta(t)$  can be determined by information up to, but not including, time  $t$ . This is not an unreasonable abstraction of "real" trading strategies. The gains process  $\int \theta dS$ , for  $\theta \in \Lambda$ , is furthermore defined path by path as a Stieltjes integral. That is, the gains at time  $t$ , are

$$\int_0^{t_i} \theta(s) dS(s) = \sum_{j=0}^{i-1} \theta(t_j) [S(t_{j+1}) - S(t_j)],$$

simply the sum of profits and losses at discrete points in time.

We will give the space  $L_Q^2$  the norm, for  $m \in L_T^2$ ,

$$\|m\|_{L_Q^2}^2 = E^*([m]_T)^{1/2}$$

and  $L_P^2[S]$  the semi-norm, for  $\theta \in L_P^2[S]$ ,  $\|\theta\|_{L_P^2[S]}^2 = E(\int_0^T \theta^2(t) d[S])$ .

**Proposition 6.1:** For every trading strategy  $\theta \in L_P^2[S]$  there exists



a sequence  $\theta_n$  of simple trading strategies converging to  $\theta$  in  $L_p^2[S]$  (in  $\|\cdot\|_{L_p^2[S]}$ ). For any such sequence, the corresponding gains processes  $\int \theta_n dS$  converge to  $\int \theta dS$  in  $\mathcal{M}_Q^2$ .

Proof: This is the way Ito originally extended the definition of stochastic integrals. His theorem uses the fact that  $\Lambda$  is dense in  $L_Q^2[S]$  and shows that  $\theta + \int \theta dS$ ,  $\theta \in \Lambda$ , extends uniquely to an isometry of  $L_Q^2[S]$  in  $\mathcal{M}_Q^2$ . These facts can be checked, for instance in Jacod [8], Chapter 4. Since  $\xi$  is assumed to be bounded above and below away from zero, the semi-norms  $\|\cdot\|_{L_p^2[S]}$  and  $\|\cdot\|_{L_Q^2[S]}$  are equivalent, and the result is proved. Q.E.D.

Interpreting this result, for any self-financing strategy  $\theta$  there is a sequence of simple trading strategies converging (as agents are able to trade more and more frequently) to  $\theta$ , with the corresponding gains processes converging to that generated by  $\theta$ . The sequence of simple strategies can be chosen to be self-financing by using the same construction shown in Section 5 for a store-of-value strategy.

In what way have we limited agents by restricting them to  $L_p^2[S]$  trading strategies? It is known, for instance, that by removing this constraint the so-called "suicide" and "doubling" strategies may become feasible.<sup>13</sup> A suicide strategy makes nothing out of something almost surely, which no one would want to do anyway. A doubling strategy, however, generates a "free lunch", which shouldn't happen in equilibrium. It can't happen for any  $L_p^2[S]$  strategies since these generate gains which are martingales (under measure Q). There is some comfort in knowing that since a doubling strategy is not in  $L_p^2[S]$ , which is a complete space, there is no sequence of simple (or even general  $L_p^2[S]$ ) strategies which converges to a doubling strategy in the manner of Proposition 6.1.



## 6.2 Some Generalizations

There is of course no difficulty in having heterogenous probability assessments, provided all subjective probability measures on  $(\Omega, \mathcal{F})$  are uniformly absolutely continuous. This preserves the topology of the consumption space across agents.

As a second generalization, we could allow the consumption space to be  $\mathbb{R} \times L^q(\mathbb{P})$  for any  $q \in [1, \infty)$ , relaxing from  $q = 2$ . The allowable trading strategies should be relaxed to  $L^q[\underline{S}]$ , as defined by Jacod [8], (4.59), since there is no guarantee of an orthogonal  $q$ -basis for  $\mathcal{M}_Q^q$ . It is a straightforward task to carry out all of the proofs in the paper under both of these generalizations. [All interesting specific models of uncertainty we are aware of are for  $q=2$ ].

It is also easy, but cumbersome, to extend all of our results to an economy with a finite number of different consumption goods and with production.

## 6.3 Economics on Event Trees

Any filtered probability space  $(\Omega, \mathcal{F}, P)$  for which  $\mathcal{F}_t$  contains a finite number of events for all  $t$  can be represented by an event tree, as in Figure 1 (after adding probabilities in the obvious manner).

For finite horizon problems, the terminal nodes of the tree can be treated as the elements of  $\Omega$ . They are equal in number with the contingent claims forming a complete regime of Arrow-Debreu "simple securities". Yet, as the following proposition demonstrates, a complete market Radner equilibrium can be established with far fewer securities (except in degenerate cases). Since integrability is not a consideration when  $\Omega$  is finite, we can characterize martingale multiplicity directly in terms of the "finite" filtration  $\mathcal{F}$ , limiting



consideration to probability measures under which each  $\omega \in \Omega$  has strictly positive probability.

Proposition 6.2: The multiplicity of a finite filtration  $\mathbf{F}$ , under any of a set of equivalent probability measures, is the maximum number of branches leaving any node of the corresponding event tree, minus one.

The proof, given in the appendix, presents a simple algorithm for constructing an orthogonal martingale basis for  $(\Omega, \mathbf{F}, Q)$ . Just as in Section 4, a complete markets Radner equilibrium exists provided there are markets for long-lived securities paying the terminal values of these orthogonal martingales (one for each) in time T consumption, and one (store-of-value) long-lived security paying one unit of consumption for each  $\omega \in \Omega$ .

By drawing simple examples of event trees, however, it soon becomes apparent that many other choices for the spanning securities will work. This is consistent with Kreps [10]. His Proposition 2 effectively states that if an Arrow-Debreu equilibrium exists, a necessary and sufficient condition for a complete markets Radner equilibrium is that at any node of the event tree the following condition is met: The dimensions of the span of the vectors of "branch-contingent" prices of the available long-lived securities must be the number of branches leaving that node. Kreps goes on to state the number of long-lived securities required for implementing an Arrow-Debreu equilibrium in this manner must be at least K, the maximum number of branches leaving any node, consistent with the "spanning number" (the multiplicity plus one) demonstrated in the previous proposition. Kreps also obtains the elegant genericity result: except for a "sparse" set of long-lived securities (a set of measure zero in a sense given in the Kreps article), any selection



of  $K$  or more long-lived security price processes admits a complete markets Radner equilibrium with the original Arrow-Debreu consumption allocations. This result seems exceedingly difficult to extend to our general continuous time model.

One should beware of taking the "limit" (by compression) of finite filtrations and expecting the spanning number to be preserved. For example, we have seen statements in the finance literature of the following sort: "In the Black-Scholes option pricing model it is to be expected that continuous trading on two securities can replicate any claim since Brownian Motion is the limit of a normalized sequence of coin-toss random walks, each of which has only two outcomes at any toss." If this logic is correct; it hides some unexplained reasoning. For example, two simultaneous independent coin-toss random walks generate a martingale space of multiplicity three (four branches at each node, minus one), whereas the corresponding Brownian Motion limits (Williams [18], Chapter 1) generate a martingale space of multiplicity two. Somehow one dimension of "local uncertainty" is lost in the limiting procedure.

#### 6.4 A Brownian Motion Example

This subsection illustrates an infinite dimensional consumption space whose economy (under regularity conditions) has a complete markets Radner equilibrium including only two securities!

Suppose uncertainty is characterized, and information is revealed, by a Standard Brownian Motion, say  $W$ . To be precise, each  $\omega \in \Omega$  corresponds to a particular sample path chosen for  $W$  from the continuous functions on  $[0, T]$  (denoted  $C[0, T]$ ) according to the Wiener measure  $P$  on  $F$ , the (completed) Borel tribe on  $C[0, T]$ . The probability space, then,



is the completed Wiener Triple  $(\Omega, \mathcal{F}, P)$ , and the filtration is the family  $\mathcal{F} = \left\{ \mathcal{F}_t; t \in [0, T] \right\}$ , where  $\mathcal{F}_t$  is the completion of the Borel tribe on  $C[0, t]$ . For conciseness, we'll call  $(\Omega, \mathcal{F}, P)$  the "completed filtered Wiener triple." More details on this framework are given in the first chapter of Williams [18].

To construct a complete markets Radner equilibrium from a given Arrow-Debreu equilibrium, as in Section 4, we need an orthogonal 2-basis for  $\mathcal{M}_Q^2$ , where  $Q$  is an equilibrium price measure for the underlying economy. In this case we can actually show that a particular Standard Brownian Motion on  $(\Omega, \mathcal{F}, Q)$  is just such a 2-basis!

It is a well known result (e.g. [11]) that the underlying Brownian Motion  $W$  is a 2-basis for  $\mathcal{M}_P^2$ . Assuming  $Q \approx P$ , an RCLL version of the process

$$Z(t) = E\left[\frac{dQ}{dP} \mid \mathcal{F}_t\right], \quad t \in [0, T]$$

is a square integrable martingale on  $(\Omega, \mathcal{F}, P)$ , with  $E[Z(T)] = 1$ . Then by Theorem 4.1 there exists some  $\rho \in L_P^2[W]$  giving the representation:

$$Z(t) = 1 + \int_0^t \rho(s) dW(s) \quad \forall t \in [0, T] \text{ a.s.}$$

It follows from Ito's Lemma that, defining the process  $r(t) = \rho(t)/Z(t)$ , we have the alternative representation:

$$Z(t) = \exp \left\{ \int_0^t r(s) dW(s) - \frac{1}{2} \int_0^t r^2(s) ds \right\} \quad \forall t \in [0, T] \text{ a.s.}$$

From this, the new process

$$W^*(t) = W(t) - \int_0^t r(s) ds, \quad t \in [0, T] \text{ a.s.} \quad (6.1)$$

defines a Standard Brownian Motion on  $(\Omega, \mathcal{F}, Q)$  by Girsanov's Fundamental Theorem (Liptser and Shirayev [12], P.232). It remains to show that  $W^*$  is itself a 2-basis for  $\mathcal{M}_Q^2$ , but this is immediate from Theorem 5.18 of Liptser and Shirayev [12], using the uniform absolute continuity of  $P$  and  $Q$ . This construction is summarized as follows.<sup>14</sup>



Proposition 6.3: Suppose  $W$  is the Standard Brownian Motion underlying the filtered Wiener triple  $(\Omega, \mathcal{F}, P)$  and  $Q \approx P$ . Then  $\overset{*}{W}$  defined by (6.1) is a Standard Brownian Motion on  $(\Omega, \mathcal{F}, Q)$  which is a 2-basis for  $\mathcal{M}_Q^2$ . In particular,  $M(\mathcal{M}_P^2) = M(\mathcal{M}_Q^2)$ .

In short, by marketing just two long-lived securities, one paying  $\overset{*}{W}(T)$  in time  $T$  consumption, the other paying one unit of time  $T$  consumption with certainty, and announcing their prices as their conditional expected consumption payoffs under  $Q$  ( $\overset{*}{W}(t)$  and 1 respectively at time  $t$ ), a complete markets Radner equilibrium is achieved.

This example can be extended to filtrations generated by vector diffusion processes. Under well known conditions (see, for example, [4]) a vector diffusion generates the same filtration as the underlying vector of independent Standard Brownian Motions. An orthogonal 2-basis for  $\mathcal{M}_P^2$  is then simply these Brownian Motions themselves ([10]). By generalizing the quoted result from Liptser and Shirayev [12] (Theorem 5.18), one can then demonstrate a vector of (equally many) Brownian Motions on  $(\Omega, \mathcal{F}, Q)$  which generates every element of  $\mathcal{M}_Q^2$  as in Theorem 4.1. Since the manipulations are rather involved, and the results raise some provocative issues concerning the "inter-temporal capital asset pricing models" (e.g. [14]) which are also based on diffusion uncertainty, we put off this development to a subsequent paper.

## 7. Concluding Remarks

We are working on several extensions and improvements suggested by the results of this paper.



The first major step will be to demonstrate the existence of continuous trading Radner equilibria "from scratch", that is taking endowments and preferences as agent primitives and proving the existence of an equilibrium such as that demonstrated in Theorem 4.1. In particular the existence of an Arrow-Debreu equilibrium and the property  $Q \approx P$  must be proven from exogenous conditions, rather than assumed. A full-blown Radner economy is also being examined, one with consumption occurring over time rather than at two points, 0 and T.

The Brownian Motion example of Section 6, as suggested there, is being extended to the case where uncertainty is characterized by a vector of diffusion "state-variable" processes. This will allow us to tie in with, and provide a critical re-evaluation of, the inter-temporal capital asset pricing models currently in vogue in the financial economics literature.

We left off in Section 5 by characterizing the spanning number in terms of (endogenous) Arrow-Debreu prices, through the equilibrium price measure Q. Our next efforts will be directed at showing that, subject to regularity conditions, martingale multiplicity is invariant under substitution of equivalent probability measures, allowing us to gain the spanning number exogenously as  $M(\mathcal{M}_P^2) + 1$ .



### Appendix on Martingale Multiplicity

What follows is a heavily condensed treatment, taken mainly from Chapter 4 of Jacod [8].

A square-integrable martingale on the filtered probability space  $(\Omega, \mathcal{F}, P)$  is an RCLL  $\mathcal{F}$ -adapted<sup>15</sup> process  $X = \{X_t; t \in [0, T]\}$  with the properties:

- (i)  $E[X(t)^2] < \infty$  for all  $t \in [0, T]$ , and
- (ii)  $E[X(t) | \mathcal{F}_s] = X(s)$  a.s. for all  $t \geq s$ .

The first property (i) is "square-integrability", the second (ii) is "martingale", meaning roughly that "the expected future value of  $X$  given current information is the current value of  $X$ ."

The space of square-integrable martingales on  $(\Omega, \mathcal{F}, P)$  which are null at zero ( $X(0) = 0$ ) is denoted  $\mathcal{M}_P^2$ . The spaces  $\mathcal{M}_P^2$  and  $L^2(P)$  are in one-to-one correspondence via the relationship between some  $x \in \mathcal{M}_P^2$  and  $x \in L^2(P)$ :

$$X(t) = E[x | \mathcal{F}_t] \quad t \in [0, T],$$

where all RCLL versions of the conditional expectation are indistinguishable, and therefore identified.

An  $\mathcal{F}$ -adapted process is termed predictable if it is measurable with respect to the tribe  $\mathcal{P}$  on  $\Omega \times [0, T]$  generated by left-continuous  $\mathcal{F}$ -adapted processes. At an intuitive level,  $\theta$  is a predictable process if the value of  $\theta(t)$  can be determined from information available up to, but not including, time  $t$ , for any  $t \in [0, T]$ .

Two martingales  $X$  and  $Y$  are said to be orthogonal if the product  $XY$  is a martingale. From this point we'll assume that  $\mathcal{F}$  is a separable tribe



under P. In that case the path breaking work of Kunita and Watanabe [11] shows the existence of an orthogonal 2-basis for  $\mathcal{M}_P^2$ , defined as a minimal set of mutually orthogonal elements of  $\mathcal{M}_P^2$  with the representation property stated in Theorem 4.1. By "minimal", we mean that no fewer elements of  $\mathcal{M}_P^2$  have this property. The number of elements of a 2-basis, whether countable infinite or some positive integer, is called the multiplicity of  $\mathcal{M}_P^2$ , denoted  $M(\mathcal{M}_P^2)$ .

The following lemma makes a technical argument used in the proof of Theorem 4.1.

Lemma A.1: Suppose the process X is defined by

$$X(t) = \sum_{n=1}^N \left[ \int_0^T \theta_n(s) dS_n(s) - \theta_n(s) S_n(s) \right], \quad t \in [0, T],$$

where  $\int \theta_n dS_n$  is a stochastic integral of a predictable process  $\theta_n$  with respect to a semi-martingale  $S_n$ , for  $1 \leq n \leq N < \infty$

Then X is predictable.

Proof: For any left-limits process Z, let  $Z(t-)$  denote the left limit of Z at  $t \in [0, T]$ , and the "jump" of Z at t by  $\Delta Z(t) = Z(t) - Z(t-)$ , where we have used the convention that  $Z(0-) = Z(0)$ .

Then we can write

$$X(t) = \sum_{n=1}^N \left[ \int_0^{t-} \theta_n(s) dS_n(s) - \theta_n(t) S_n(t) + \theta_n(t) \Delta S_n(t) \right]$$

since  $\Delta \left( \int_0^t \theta_n(s) dS_n(s) \right) = \theta_n(t) \Delta S_n(t)$  by the definition of a stochastic integral. Then, using  $\theta_n(t) S_n(t) = \theta_n(t) [S_n(t-) + \Delta S_n(t)]$ ,

we have

$$X(t) = \sum_{n=1}^N \left[ \int_0^{t-} \theta_n(s) dS_n(s) - \theta_n(t) S_n(t-) \right], \quad t \in [0, T].$$

Since  $\int_0^{t-} \theta_n dS_n$  and  $S_n(t-)$  are left-continuous processes, and therefore



predictable, and  $\theta_n(t)$  is predictable, we know  $X$  is predictable since products and sums of measurable functions are measurable. Q.E.D.

For any two elements  $X$  and  $Y$  of  $\mathcal{M}_P^2$ , let  $\langle X, Y \rangle$  denote the unique predictable process with the property that  $XY - \langle X, Y \rangle$  is a martingale and  $\langle X, Y \rangle_0 = 0$ .

Lemma A.2: Suppose  $(m_1, \dots, m_N)$  are elements of  $\mathcal{M}_Q^2$  with the representation property given in the statement of Theorem 4.1, where  $N = M(\mathcal{M}_Q^2) < \infty$ . If  $\theta_n$  and  $\phi_n$  are elements of  $L_Q^2[m_n]$ ,  $1 \leq n \leq N$  satisfying (with the obvious shorthand)

$$\int_0^t \theta_s^\top dm_s = \int_0^t \phi_s^\top dm_s \quad \forall t \in [0, T] \text{ a.s.} \quad (\text{a.1})$$

then

$$Q\left\{\exists t \in [0, T] : \theta(t) = \phi(t)\right\} > 0.$$

Proof: Jacod [8] shows the existence of a predictable positive semi-definite  $N \times N$  matrix valued process  $c$  and an increasing predictable process  $C$  with the property, for any  $\alpha_n, \beta_n \in L_Q^2[m_n]$ ,  $1 \leq n \leq N$ ,

$$\langle \int \alpha^\top dm, \int \beta^\top dm \rangle_t = \int_0^t \alpha(s)^\top c(s) \beta(s) dC(s) \quad \forall t \in [0, T] \text{ a.s.} \quad (\text{a.2})$$

The process  $C$  also defines a Doleans measure (also denoted  $C$ ) on  $(\Omega \times [0, T], \mathcal{P})$  according to

$$C(B) = \int_{\Omega} \int_{[0, T]} 1_B(\omega, s) dC(\omega, s) Q(d\omega) \quad \forall B \in \mathcal{P}.$$

By (4.43) of Jacod [7], the matrix process  $c$  reaches full rank, and is thus positive definite, on some set  $B^* \in \mathcal{P}$  of strictly positive  $C$  measure. But, by (a.1) and (a.2),

$$\begin{aligned} \int_0^t [\theta(w, s) - \phi(w, s)]^\top c(w, s) [\theta(w, s) - \phi(w, s)] dC(w, s) \\ = 0 \quad \forall t \in [0, T] \text{ a.s.} \quad (\text{a.3}) \end{aligned}$$



Ignoring without loss of generality the  $Q$ -null set on which (a.3) does not hold, this implies that  $\theta(w,t) = \phi(w,t)$  for all time points of increase of  $C$  on  $B^*$ , which have strictly positive  $Q$ -probability since the projection of  $B^*$  on  $\Omega$  must have strictly positive  $Q$  measure to have strictly positive  $C$ -measure. Q.E.D.

Proof of Proposition 6.2:

Let  $N$  denote the maximum number of branches leaving any node of the event tree minus one. The proposition will be proved by constructing an orthogonal martingale basis for the space of martingales on this filtration consisting of  $N$  processes,  $m_1, \dots, m_N$ . Any martingale on a finite filtration is determined entirely by its right-continuous jumps at each node in the corresponding event tree. Denote the jump of  $m_j$  at a generic node with  $L$  departing branches by the vector  $\delta_j = (\delta_{j1}, \dots, \delta_{jL})$ . That is,  $\delta_j \in \mathbb{R}^L$  represents the random variable which takes the real number  $\delta_{j1}$  if branch 1 is the realized event at this node. Let  $p = (p_1, \dots, p_L) \in \mathbb{R}^L$  denote the vector of conditional branching probabilities at this node.

The processes  $m_1, \dots, m_N$  are then mutually orthogonal martingales if they satisfy the following two conditions at each node:

- (i)  $p^T \delta_j = 0, j = 1, 2, \dots, N$  (zero mean jumps, the martingale property), and
- (ii)  $\delta_j^T [p] \delta_k = 0 \quad \forall j \neq k$ , where  $[p]$  denotes the diagonal matrix whose 1-th diagonal element is  $p_1$  (mutually uncorrelated jumps, implying mutually orthogonal martingales).

We construct the processes  $m_1, \dots, m_N$  by designing their jumps at each node of the event tree, in any order, taking  $m_j(0) = 0 \quad \forall j$ . At a given node (with  $L$  branches), it is simple to choose non-zero



vectors  $\delta_1, \dots, \delta_{L-1}$  in  $R^L$  satisfying

$$\Delta_j[p] \delta_j = 0 \quad j = 1, \dots, L-1, \quad (a.4)$$

where  $\Delta_j$  is a  $j \times L$  matrix whose first row is a vector of ones and

whose  $k$ -th row is  $\delta_{k-1}^\top$ . This cannot be done for  $j \geq L$  if

$\Delta_L[p]^{1/2}$  is a full rank  $L \times L$  matrix (its rows are non-zero and mutually orthogonal). Instead, let  $\delta_L, \dots, \delta_N$  each be zero

vectors. One can quickly verify that this construction meets the conditions (i)-(ii) for  $m_1, \dots, m_N$  to be mutually orthogonal martingales. They are non-trivial since there is at least one node with  $N+1$  branches. They form a basis (in the sense of stochastic integration) for all martingales since at each node (with the obvious notation) the subspace  $\{ \delta \in R^L : \delta^\top p = 0 \}$  has the linearly independent spanning vectors  $(\delta_1, \dots, \delta_{L-1})$ . (That is, the jump of any given martingale at this node is a linear combination of the jumps of those martingales in  $(m_1, \dots, m_N)$  "active" at this node.) At least  $N$  martingales are needed for a martingale basis since at some node this subspace has dimension  $N$ , by definition of  $N$ .

Q.E.D.

As a simple example, consider a finite-state space Markov chain information structure. Transition possibilities given by the matrix

$$\Pi = (\pi_{\alpha\beta}) \quad \begin{matrix} \alpha = 1, \dots, n \\ \beta = 1, \dots, n \end{matrix},$$

where  $\pi_{\alpha\beta}$  is the probability of one-step transition from state  $\alpha$  to state  $\beta$ . Let  $\pi^\alpha$  denote the  $\alpha$ -th row of  $\Pi$  and  $\delta_j^\alpha \in R^n$  the vector of jumps of the process  $m_j$  at any node correspondint to state  $\alpha$ ,  $j = 1, \dots, n-1$ . We'll assume at least one row of  $\Pi$  has no zero elements. Then the multiplicity of the space of martingales on this Markov chain is  $n-1$ , and  $m_1, \dots, m_{n-1}$  is an orthogonal martingale basis provided, for  $\alpha = 1, \dots, n$ ,



$$\pi^{\alpha T} \delta_j^\alpha = 0 \quad j = 1, \dots, n-1$$

and  $\delta_j^{\alpha T} [\pi^\alpha] \delta_k^\alpha \quad j \neq k,$

corresponding to conditions (i) - (ii) above, and  $\delta_j^\alpha \neq 0$

$\forall \alpha, \forall j.$  If, for instance,

$$\Pi = \begin{bmatrix} .3 & .3 & .4 \\ .3 & .3 & .4 \\ .3 & .3 & .4 \end{bmatrix},$$

then the two martingales  $m_1$  and  $m_2$  are an orthogonal martingale basis, where, at any node,  $m_1$  jumps +2 if state 1 occurs at the next step, +2 if state 2 occurs, and -3 if state 3 occurs, or  $\delta_1 = (2, 2, -3)$ ; and  $\delta_2 = (1, -1, 0)$  describes the state contingent jumps of  $m_2$  at any node. To further illustrate, if state 2 occurs at time 1, state 3 at time 2, and the chain terminates at time 2.5, the sample path for  $m_1$  is

$$\begin{aligned} m_1(t) &= 0 & , & 0 \leq t < 1 \\ m_1(t) &= 2 & , & 1 \leq t < 2 \\ m_1(t) &= -1 & , & 2 \leq t \leq 2.5. \end{aligned}$$



NOTES

1. Merton [15] is also seminal in this regard. Similar results were obtained by Cox and Ross [2] for other models of uncertainty.
2. This is sometimes misleadingly interpreted as the "state of information at time  $t$ ", whereas it is more reasonably treated as the set of all possible states of information at time  $t$ .
3. While it is conventional to assume that consumption allocations and endowments are restricted to be positive (elements of the positive cone of  $V$ ), it is a matter of indifference to our model whether or not this restriction is adopted.
4. See Schaefer [17].
5. Sufficient conditions for this can be given when preferences can be represented by Von Neumann-Morgenstern utility functions, in terms of bounds on marginal utilities for time  $T$  consumption. We do not pursue this here since we are taking  $\xi$  as a primitive, rather than deriving it from preferences.
6. A tribe  $F$  is said to be separable under  $P$  if there exists a countable number of elements  $B_1, B_2, \dots$  in  $F$  such that, for any  $B \in F$  and  $\epsilon > 0$  there exists  $B_n$  with  $P\{B \Delta B_n\} < \epsilon$ , where  $\Delta$  denotes symmetric differences.
7. See, for example, Jacod [8] for the definition of a semi-martingale. This is not in the least restrictive.
8. Memin [13] lists sufficient conditions, but in this paper condition (ii) is sufficient for (iii), so we do not need to repeat that list here.
9. By "RCLL", we mean a process whose sample paths are almost surely right-continuous with left limits.
10. Note that  $E^*(x | F_T) = x$  a.s.
11. This is easily done for event trees. From this proof of the Proposition it is apparent that a selection of consumption payoffs for long-lived securities can be designed which (generically) completes markets for any Arrow-Debreu equilibrium prices.
12. These are questions raised by Harrison and Pliska [6].
13. See Harrison and Pliska [6], p.250 and Kreps [9].
14. By a slightly more subtle argument, we could have reached the same conclusions under the weaker assumption that  $P$  and  $Q$  are merely equivalent, but  $Q \not\cong P$  is needed for other reasons in Theorem 4.1.
15. A process  $X = \{X_t, t \in [0, T]\}$  is adapted to  $F = \{F_t; t \in [0, T]\}$  if  $X_t$  is measurable with respect to  $F_t$  for all  $t \in [0, T]$ .



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